Assignment 12.

This homework is due *Thursday*, Dec 3.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

This homework is somewhat long, so it will have weight 1.5 compared to a usual homework.

(1) (9.3.36) Define the function $\psi : C[a,b] \to \mathbb{R}$ by

$$\psi(f) = \int_a^b f(x)dx \quad \text{for each } f \in C[a,b].$$

Show that ψ is Lipschitz on $C[a,b]$ with the metric induced by the maximum norm $\|\cdot\|_\infty$.

(2) (9.5.64) Let X be a totally bounded metric space. If f is a uniformly continuous mapping from X to a metric space Y, show that $f(X)$ is totally bounded. Is the same true if f is only required to be continuous?

(3) (9.5.55, Baby Tychonoff's theorem) Prove that if metric spaces (X,ρ) and (Y,σ) are compact, then so is $X \times Y$ with product metric.

(4) (9.5.69) For a compact metric space (X,ρ), show that there are points $u, v \in X$ for which $\rho(u,v) = \operatorname{diam} X$.
(Reminder: $\operatorname{diam} X = \sup\{\rho(x,y) \mid x, y \in X\}$.)

(5) (10.3.33) Let (X,ρ) be a *compact* metric space and T a mapping $X \to X$ such that

$$\rho(T(u),T(v)) < \rho(u,v) \text{ for all } u \neq v \in X.$$

Show that T has a unique fixed point. (*Hint: Option 1*: Use Extreme Value theorem directly. *Option 2*: Show that if there are no fixed points, the function $\rho(T(u),T^2(u))/\rho(u,T(u))$ from X to $\mathbb{R}$ is continuous and therefore reaches its maximum. Then follow the proof of Banach Contraction Principle using Problem 7 of HW11.)

The problems below can be found in the Section 10.2 of textbook.

(6) (a) Recall that in Problem 3 of Homework 11 we defined interior $\operatorname{int} E$, exterior $\operatorname{ext} E$ and boundary $\operatorname{bd} E$ of a subset E of a metric space. Show that for every subset E of a metric X, $X = \operatorname{int} E \cup \operatorname{ext} E \cup \operatorname{bd} E$ and the union is disjoint.

(b) Recall that a subset A of a metric space X is called *dense* in X if every nonempty open subset of X contains a point of A. Further, a subset of a metric space X is called hollow in X if it has empty interior. Show that for a subset E of a metric space X, E is hollow in X if and only if $X \setminus E$ is dense in X.

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(7) Prove the following theorem:
(*The Baire Category Theorem.*) Let X be a complete metric space. Let $\{\mathcal{O}_n\}$ be a countable collection of open dense subsets of X. Then the intersection $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ also is dense.
(*Hint:* You need to show that an arbitrary open ball $B(x_0, r_0)$ contains a point of $\bigcap_{n=1}^{\infty} \mathcal{O}_n$. Start by saying that $B(x_0, r_0) \cap \mathcal{O}_1$ is nonempty (why) and open (why), therefore contains an open ball $B(x_1, r_1)$ and a smaller closed[1] ball $B_1 = \overline{B}(x_1, r_1/2)$. Repeat argument with the open ball $B(x_1, r_1/2)$ and $\mathcal{O}_2$, and so on. Get a descending sequence of closed balls $B_1, B_2, \ldots$. Apply the Cantor Intersection Theorem.)

(8) Prove the following theorem:
(*The Baire Category Theorem.*) Let X be a complete metric space. Let $\{F_n\}$ be a countable collection of closed hollow subsets of X. Then the union $\bigcup_{n=1}^{\infty} F_n$ is also hollow.
(*Hint:* Apply Problem 6b to the assertion of Problem 7.)

(9) Let X be a complete metric space and $\{F_n\}$ a countable collection of closed subsets of X. If $\bigcup_{n=1}^{\infty} F_n$ has nonempty interior (for example, if $\bigcup_{n=1}^{\infty} F_n = X$), prove that at least one of the F_n's has nonempty interior.
(*Hint:* Pass to appropriate closed subset of X. Use Problem 8.)
The above result is also called Baire Category Theorem.

(10) Prove the following theorem.
Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X that is pointwise bounded in the sense that for each $x \in \mathbb{X}$, there is a constant M_x for which

$$|f(x)| \leq M_x \text{ for all } f \in \mathcal{F}.$$

Then there is nonempty open subset $\mathcal{O}$ of $\mathbb{X}$ on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant M for which

$$|f| \leq M \text{ on } \mathcal{O} \text{ for all } f \in \mathcal{F}.$$

(*Hint:* Define $E_n = \{x \in X : \ |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$. Use Problem 9.)

1. Extra Problems

(11) (10.2.20) Let F_n be the subset of $C[0,1]$ consisting of functions for which there is a point x_0 in $[0,1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.
  (a) Show that F_n is closed.
  (b) Show that F_n is hollow. (*Hint:* Show that for $f \in C[0,1]$ and $r > 0$, there is a piecewise linear "saw-like" function $g \in C[0,1]$ for which $\rho_\infty(f, g) < r$ and the left-hand and right-hand derivatives of g on $[0,1]$ are greater than $n+1$.)
  (c) Conclude by Baire Category theorem that $C[0,1] \neq \bigcup_{n=1}^{\infty} F_n$.
  (d) Show that each $h \in C[0,1] \setminus \bigcup_{n=1}^{\infty} F_n$ is not differentiable at any point in $[0,1]$. (*Hint:* If f is differentiable at x_0 and continuous on $[0,1]$, then $|f(x) - f(x_0)|/|x - x_0|$ is bounded "close" to x_0 by differentiability, and bounded "far" from x_0 by boundedness of f on $[0,1]$; so it belongs to some F_n.)

Note. Congratulations, you proved that there are continuous functions on $[0,1]$ that are not differentiable *anywhere*. Moreover, you proved that the set of such functions is *dense* in $C[0,1]$.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and have derivatives of all orders. Suppose that for each $x \in \mathbb{R}$, there is index $n = n(x)$ for which $f^{(n)}(x) = 0$. Show that f is a polynomial. (*Hint:* Use Baire Category Theorem.)
Comment. If you know that n is the same for all x, the statement easily follows by calculus.

[1]A closed ball $\overline{B}(x, r)$ is the set $\{y \in X \mid \rho(x, y) \leq r\}$. It is a closed set.